

A trapped surface in the higher-dimensional self-similar Vaidya spacetime

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We investigate a trapped surface and naked singularity in a D -dimensional Vaidya spacetime with a self-similar mass function. A trapped surface is defined as a closed spacelike $(D-2)$ -surface which has negative both null expansions. There is no trapped surface in the Minkowski spacetime. However, in a four-dimensional self-similar Vaidya spacetime, Bengtsson and Senovilla considered non-spherical trapped surfaces and showed that a trapped surface can penetrate into a flat region, if and only if the mass function rises fast enough [I. Bengtsson and J. M. M. Senovilla, Phys. Rev. D **79**, 024027 (2009).]. We apply this result to a D -dimensional spacetime motivated by the context of large extra dimensions or TeV-scale gravity. In this paper, similarly to Bengtsson and Senovilla's study, we match four types of $(D-2)$ -surfaces and show that a trapped surface extended into the flat region can be constructed in the D -dimensional Vaidya spacetime, if the increasing rate of the mass function is greater than 0.4628. Moreover, we show that the maximum radius of the trapped surface constructed here approaches the Schwarzschild-Tangherlini radius in the large D limit. Also, we show that there is no naked singularity, if the spacetime has the trapped surface constructed here.

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I. INTRODUCTION

The boundary of a region in a spacetime that cannot be observed from infinity is called an event horizon. The event horizon is the region of the boundary of a black hole which has a teleological property: the entire future history of the spacetime must be known before the position of the event horizon can be determined. Black holes might be formed by some dynamical process, and then might undergo accretions and evolutionary processes. In numerical relativity, to investigate the evolution of the black hole we need to identify the boundary of the black hole in an initial data set. From this context it is difficult to investigate the evolution of the black hole by using the event horizon in numerical relativity.

Eardley conjectured that the boundary of the region which contains marginally outer trapped surfaces coincides with the event horizon [1], where an outer trapped surface is defined as a closed spacelike two-surface (in the four-dimensional case) which has a negative outer null expansion. Although the event horizon has the teleological notion and is defined in terms of future null infinity, Eardley's conjecture suggests that the event horizon can be constructed by the outer trapped surface without the teleological notion. In a four-dimensional Vaidya spacetime, Ben-Dov showed that Eardley's conjecture is true [2]. To study this conjecture in various spacetimes might be important to understand dynamical black holes. However, the outer trapped surface cannot be defined in general spacetimes, while it is defined only in asymptotically flat spacetimes [3]. Moreover, the outer trapped surface is only defined by outgoing null rays. We do not know whether the outer trapped surface has a negative ingoing null expansion or not. To resolve this difficulty we consider a trapped surface, where a trapped surface is defined as the closed spacelike two-surface which has negative both null expansions. In general spacetimes, while ingoing and outgoing null rays are not defined, there are two independent null rays. Both null expansions are defined by these two independent null rays. Thus, the trapped surface can be defined not only in asymptotically flat spacetimes but also in various spacetimes.

As is well known, there is no trapped surface in the Minkowski spacetime. However, in the four-dimensional Vaidya spacetime it was shown that a trapped surface can penetrate into a flat region. The numerical results of Schnetter and Krishnan showed that the outer boundary of trapped surfaces can penetrate into the flat region [4]. Moreover, Bengtsson and Senovilla considered the self-similar Vaidya spacetime, and they analytically showed that a trapped surface can penetrate into the flat region, if and only if the mass function rises fast enough [5]. Is this feature common

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in higher-dimensional spacetimes? In the present paper, we focus on a higher-dimensional Vaidya spacetime and investigate a trapped surface in this spacetime.

Recently, higher-dimensional scenarios with large [6] and warped [7] extra dimensions were proposed to resolve the hierarchy problem between the gravitational and electroweak interactions. One of the most striking predictions of such scenarios is the production of the large number of mini black holes in high-energy particle collisions [8]. Therefore, to study higher-dimensional black holes is important in the context of above scenarios.

Can we construct a trapped surface extended into the flat region in the higher-dimensional spacetime as in the four-dimensional spacetime? In the present paper, we apply Bengtsson and Senovilla's study to an $(n+3)$ -dimensional ($n = D - 3 \geq 1$) Vaidya spacetime with a self-similar mass function. We concern a trapped surface constructed by matching four types of $(n+1)$ -surfaces and show that a trapped surface can penetrate into the flat region. Moreover, we demonstrate that there is no naked singularity, if the spacetime has the trapped surface constructed here.

This paper is organized as follows. In Sec. II, we briefly review the $(n+3)$ -dimensional self-similar Vaidya spacetime and mention the condition for the black hole and naked singularity in this spacetime. In Sec. III, we introduce two classes of $(n+1)$ -surfaces to construct a trapped surface, and calculate both null expansions of these surfaces. In Sec. IV, we consider four types of $(n+1)$ -surfaces and match these surfaces. Then, we construct a trapped surface extended into the flat region and show the condition to construct this surface. Moreover, we discuss the relation of the condition for the trapped surface and naked singularity, and demonstrate that there is no naked singularity if the spacetime has the trapped surface constructed here. We conclude the paper in Sec. V.

II. $(n+3)$ -DIMENSIONAL VAIDYA SPACETIME

We focus on the $(n+3)$ -dimensional ($n = D - 3 \geq 1$) Vaidya spacetime [9]

$$ds^2 = - \left(1 - \frac{2m(v)}{nr^n} \right) dv^2 + 2dvdr + r^2 d\Omega_{n+1}^2, \quad (2.1)$$

where $m(v)$ is the mass function, and

$$d\Omega_{n+1}^2 = d\theta_1^2 + \sin^2 \theta_1 d\theta_2^2 + \cdots + \left(\prod_{i=1}^n \sin^2 \theta_i \right) d\varphi^2 \quad (2.2)$$

is the line element on an unit $(n+1)$ -sphere. θ_i is an inclination angle defined in $0 \leq \theta_i \leq \pi$ and φ is an azimuthal angle defined in $0 \leq \varphi \leq 2\pi$, respectively. For an arbitrary $m(v)$, the following stress energy tensor solves the Einstein equation,

$$T_{\mu\nu} = \frac{(n+1)\dot{m}}{8\pi} l_\mu l_\nu, \quad (2.3)$$

where $l_\mu = -\nabla_\mu v$ is an ingoing null vector, ∇_μ is the covariant derivative in the $(n+3)$ -dimensional spacetime, and the dot is the differentiation with respect to v . In the present paper, we choose the mass function such as

$$m = \begin{cases} 0 & \text{for } v \leq 0 \\ \mu v^n & \text{for } 0 \leq v \leq v_0 \\ M & \text{for } v \geq v_0 \end{cases}, \quad (2.4)$$

where $v_0 = M^{1/n}/\mu$, μ and M are positive constants, respectively. We call μ a *mass parameter*. There is the radial influx of a null fluid for an initially empty region. We call the region $v < 0$ *region I*. The region $0 \leq v \leq v_0$ has the radial influx. We call this region *region II*. The region $v \geq v_0$ has a constant mass is called *region III*. These regions are shown in Fig. 1. Note that naked singularity occurs if and only if the mass parameter satisfies the following condition [10]:

$$0 < \mu \leq \mu_c = \left\{ \frac{n}{2(n+1)} \right\}^{n+1}. \quad (2.5)$$

For $\mu > \mu_c$, singularity is not even naked, i.e., the spacetime has the black hole. Conformal diagrams of these spacetimes are drawn in Fig. 1.

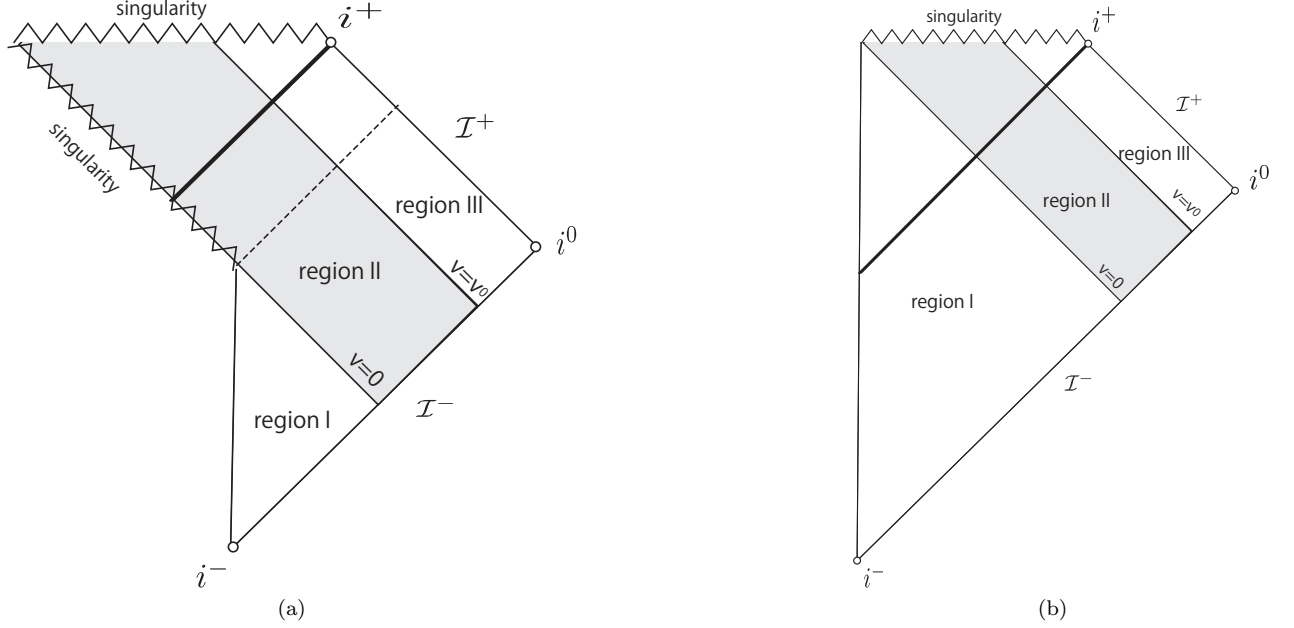


FIG. 1: Conformal diagrams of the $(n+3)$ -dimensional Vaidya spacetime. The shaded region is region II. This region is bounded by region I on $v=0$ and region III on $v=v_0$. (a) The spacetime collapses to globally naked singularity, where the mass parameter satisfies $0 < \mu \leq \mu_c$. The event horizon and the Cauchy horizon are shown as the bold and the dashed lines. (b) The spacetime collapses to the black hole, where the mass parameter satisfies $\mu > \mu_c$. The thick solid line is the event horizon.

III. BOTH NULL EXPANSIONS FOR TWO CLASSES OF $(n+1)$ -SURFACES

In Bengtsson and Senovilla's study they concerned the four-dimensional self-similar Vaidya spacetime and used two classes of two-surfaces to construct a trapped surface extended into the flat region. To apply this study to the case of the $(n+3)$ -dimensional spacetime we introduce two classes of $(n+1)$ -surfaces as in Table 1. We call the surface in

TABLE I: The class of $(n+1)$ -surfaces where $i, j = 1, \dots, n$

class	v	r	θ_i	$\theta_{j \neq i}$	φ
A	$V(\rho)$	$R(\rho)$	$\pi/2$	$0 \leq \theta_j \leq \pi$	$0 \leq \varphi \leq 2\pi$
B	$V(\rho)$	r_0	$\Theta_i(\rho)$	$0 \leq \theta_j \leq \pi$	$0 \leq \varphi \leq 2\pi$

which r and v are the function of ρ , and $\theta_i = \pi/2$ class A, where $i = 1, \dots, n$. Similarly, we call the surface in which θ_i and v are the function of ρ , and $r = r_0$ class B, where $i = 1, \dots, n$. We shall calculate both null expansions for these two classes of $(n+1)$ -surfaces.

A. Both null expansions for a class A surface

We introduce the following the $(n+1)$ -surface of class A:

$$\theta_1 = \frac{\pi}{2}, \quad r = R(\rho), \quad v = V(\rho), \quad \theta_2 = \phi_2, \quad \dots, \quad \theta_n = \phi_n, \quad \varphi = \phi_{n+1}, \quad (3.1)$$

where we have chosen $i = 1$. The first fundamental form on this surface is

$$d\gamma^2 = \Delta d\rho^2 + R^2 d\Omega_n^2, \quad (3.2)$$

where $d\Omega_n^2$ is the line element on an unit n -sphere, we have put

$$\Delta = V' (2R' + \Sigma V'), \quad (3.3)$$

and

$$\Sigma = \frac{2m}{nR^n} - 1, \quad (3.4)$$

and the prime denotes the differentiation with respect to ρ . To obtain a spacelike $(n+1)$ -surface we demand $\Delta > 0$, and hence we must have $V' \neq 0$. We choose orthonormal basis vectors tangent to this $(n+1)$ -surface as follows:

$$\begin{aligned} Y_{(1)}^\mu &= \frac{V'}{\sqrt{\Delta}} \left(\frac{\partial}{\partial v} \right)^\mu + \frac{R'}{\sqrt{\Delta}} \left(\frac{\partial}{\partial r} \right)^\mu, & Y_{(2)}^\mu &= \frac{1}{R} \left(\frac{\partial}{\partial \theta_2} \right)^\mu, \\ Y_{(J)}^\mu &= \frac{1}{R} \left(\prod_{k=2}^{J-1} \frac{1}{\sin \phi_k} \right) \left(\frac{\partial}{\partial \theta_J} \right)^\mu, & Y_{(n+1)}^\mu &= \frac{1}{R} \left(\prod_{k=2}^n \frac{1}{\sin \phi_k} \right) \left(\frac{\partial}{\partial \varphi} \right)^\mu, \end{aligned} \quad (3.5)$$

where $J = 3, \dots, n$. Also, we choose normal vectors of this $(n+1)$ -surface as

$$n_\mu = \frac{1}{\sqrt{\Delta}} \{ -R'(dv)_\mu + V'(dr)_\mu \}, \quad e_\mu = R(d\theta_1)_\mu, \quad (3.6)$$

where n_μ and e_μ are timelike and spacelike vectors, respectively. These vectors satisfy conditions $-n^2 = e^2 = 1$ and $n \cdot e = 0$. Using these normal vectors, we obtain null normals as

$$N_\mu^{(+)} = \frac{1}{\sqrt{2}} (n_\mu + e_\mu), \quad N_\mu^{(-)} = \frac{1}{\sqrt{2}} (n_\mu - e_\mu). \quad (3.7)$$

Both null expansions are given by [3]

$$\vartheta_{(\pm)} = \left(\nabla_\mu N_\nu^{(\pm)} \right) \sum_{I=1}^{n+1} Y_{(I)}^\mu Y_{(I)}^\nu. \quad (3.8)$$

Substituting orthonormal basis vectors (3.5) and null normals (3.7) into Eq. (3.8), and after some calculation, we obtain both null expansions as follows:

$$\vartheta_{(\pm)} = \frac{1}{\sqrt{2\Delta}} \left\{ \frac{1}{\Delta} \left(R'V'' - V'R'' + \frac{mV'^2R'}{R^{n+1}} - \frac{\dot{m}V'^3}{nR^n} \right) - \frac{nR'}{R} + \frac{V'}{R} \left(n - \frac{m}{R^n} \right) \right\}. \quad (3.9)$$

Note that we can introduce class A surfaces for $i \neq 1$ and can calculate these both null expansions. Then, we find that the both null expansions for these surfaces are written in the same form as Eq. (3.9).

B. Both null expansions for a class B surface

We introduce the $(n+1)$ -surface of class B

$$\theta_1 = \Theta_1(\rho), \quad v = V(\rho), \quad r = r_0, \quad \theta_2 = \phi_2, \quad \dots, \quad \theta_n = \phi_n, \quad \varphi = \phi_{n+1}, \quad (3.10)$$

where r_0 is a positive constant, and we have chosen $i = 1$. The first fundamental form is

$$d\gamma_1^2 = \Delta_1 d\rho^2 + r_0^2 \sin^2 \Theta_1 d\Omega_n^2, \quad (3.11)$$

where we have put

$$\Delta_1 = \Sigma V'^2 + r_0^2 \Theta_1'^2, \quad (3.12)$$

and Σ is written in the same form as Eq. (3.4). To obtain the spacelike $(n+1)$ -surface we also demand $\Delta_1 > 0$. We choose orthonormal basis vectors tangent to this surface as follows:

$$\begin{aligned} Y_{(1)}^\mu &= \frac{V'}{\sqrt{\Delta_1}} \left(\frac{\partial}{\partial v} \right)^\mu + \frac{\Theta_1'}{\sqrt{\Delta_1}} \left(\frac{\partial}{\partial \theta_1} \right)^\mu, & Y_{(2)}^\mu &= \frac{1}{r_0 \sin \Theta_1} \left(\frac{\partial}{\partial \theta_2} \right)^\mu, \\ Y_{(J)}^\mu &= \frac{1}{r_0 \sin \Theta_1} \left(\prod_{k=2}^{J-1} \frac{1}{\sin \phi_k} \right) \left(\frac{\partial}{\partial \theta_J} \right)^\mu, & Y_{(n+1)}^\mu &= \frac{1}{r_0 \sin \Theta_1} \left(\prod_{k=2}^n \frac{1}{\sin \phi_k} \right) \left(\frac{\partial}{\partial \varphi} \right)^\mu, \end{aligned} \quad (3.13)$$

where $J = 3, \dots, n$. On the other hand, null normals of this surface are

$$\begin{aligned} N_\mu^{(+)} &= -\frac{r_0 \Theta'_1 \sqrt{\Sigma}}{\sqrt{2\Delta_1}} (dv)_\mu + \frac{1}{\sqrt{2\Sigma}} \left(1 - \frac{r_0 \Theta'_1}{\sqrt{\Delta_1}}\right) (dr)_\mu + \frac{r_0 V' \sqrt{\Sigma}}{\sqrt{2\Delta_1}} (d\theta_1)_\mu, \\ N_\mu^{(-)} &= \frac{r_0 \Theta'_1 \sqrt{\Sigma}}{\sqrt{2\Delta_1}} (dv)_\mu + \frac{1}{\sqrt{2\Sigma}} \left(1 + \frac{r_0 \Theta'_1}{\sqrt{\Delta_1}}\right) (dr)_\mu - \frac{r_0 V' \sqrt{\Sigma}}{\sqrt{2\Delta_1}} (d\theta_1)_\mu. \end{aligned} \quad (3.14)$$

Substituting Eqs. (3.13) and (3.14) into Eq. (3.8), we obtain both null expansions of a class B surface for $i = 1$ as follows:

$$\begin{aligned} \vartheta_{1(\pm)} &= \frac{\sqrt{\Sigma}}{\sqrt{2\Delta_1}\Delta_1} \left[\pm r_0 (\Theta'_1 V'' - V' \Theta''_1) + \frac{\sqrt{\Delta_1}}{r_0} \left\{ n \left(1 - \frac{m}{nr_0^n}\right) V'^2 - (n+1) r_0^2 \Theta_1'^2 \right\} \right. \\ &\quad \left. \pm n \frac{\Delta_1 V'}{r_0} \cot \Theta_1 \right] - \frac{\dot{m} V'^2 (\sqrt{\Delta_1} \mp r_0 \Theta'_1)}{\sqrt{2\Sigma}\Delta_1 n r_0^n}. \end{aligned} \quad (3.15)$$

Also, from a similar calculation both null expansions of the class B surface for $i \neq 1$ can be obtained as follows:

$$\begin{aligned} \vartheta_{i(\pm)} &= \frac{\sqrt{\Sigma}}{\sqrt{2\Delta_i}\Delta_i} \left[\pm r_0 (\Theta'_i V'' - V' \Theta''_i) \left(\prod_{k=1}^{i-1} \sin \phi_k \right) + \frac{\sqrt{\Delta_i}}{r_0} \left\{ n \left(1 - \frac{m}{nr_0^n}\right) V'^2 - (n+1) r_0^2 \Theta_i'^2 \left(\prod_{k=1}^{i-1} \sin^2 \phi_k \right) \right\} \right. \\ &\quad \left. \pm n \frac{\Delta_i V'}{r_0} \cot \Theta_i \left(\prod_{k=1}^{i-1} \frac{1}{\sin \phi_k} \right) \right] - \frac{\dot{m} V'^2 (\sqrt{\Delta_i} \mp r_0 \Theta'_i)}{\sqrt{2\Sigma}\Delta_i \Delta_i n r_0^n}, \end{aligned} \quad (3.16)$$

where $i = 2, \dots, n$, $\theta_i = \Theta_i(\rho)$, and we have put

$$\Delta_i = \Sigma V'^2 + r_0^2 \Theta_i'^2 \prod_{k=1}^{i-1} \sin^2 \phi_k. \quad (3.17)$$

We also have demanded $\Delta_i > 0$ to obtain the spacelike surface.

IV. CONSTRUCTION OF A TRAPPED SURFACE

In this section, using surfaces of class A and B, we present four types of $(n+1)$ -surfaces and construct a trapped surface extended into region I. In region I, we introduce the $(n+1)$ -surface which is one of class A surfaces and is a topological disk given by the hyperboloid. We call this surface *type A1*. In region II, we introduce the $(n+1)$ -surface which is also of class A and call this surface *type A2*. In region III, we consider two types of $(n+1)$ -surfaces. The first one has the property of classes A and B, and we call this surface *type AB*. The second is class B and is called *type B1*. These types of $(n+1)$ -surfaces are shown in Fig. 2. We find the condition for each type of $(n+1)$ -surfaces to have negative both null expansions. We also find the condition for these four types of $(n+1)$ -surfaces to consist a smooth closed $(n+1)$ -surface. We take an intersection of these conditions and discuss naked singularity.

A. four types of $(n+1)$ -surfaces and the condition for negative both null expansions

1. Type A1 surface

In region I, we introduce the $(n+1)$ -surface of class A which is the topological disk given by the hyperboloid

$$V = t_0 + \rho - \sqrt{\rho^2 + k^2}, \quad R = \rho \quad (4.1)$$

where t_0 and k are positive constants, and $t_0 < k$. We call this surface *type A1*. In this surface, we find

$$V' = 1 - \frac{R}{\sqrt{R^2 + k^2}}, \quad V'' = -\frac{k^2}{(R^2 + k^2)^{3/2}}, \quad R' = 1 \quad \text{and} \quad R'' = 0. \quad (4.2)$$

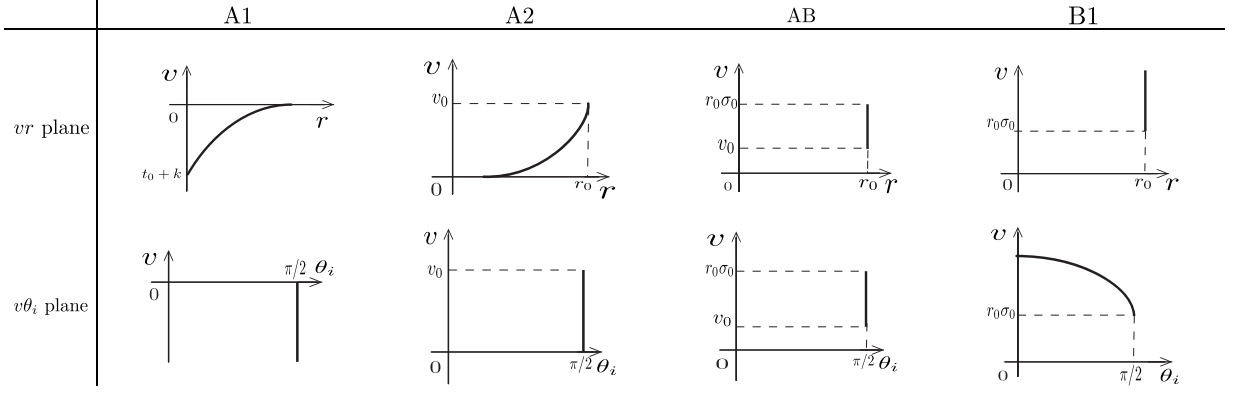


FIG. 2: Four types of $(n+1)$ -surfaces on vr and $v\theta_i$ planes.

Substituting Eq. (4.2) and $m = 0$ into Eq. (3.9), we obtain negative both null expansions as follows:

$$\vartheta_{(\pm)} = -\frac{(n+1)}{\sqrt{2}k} < 0. \quad (4.3)$$

As noted in Sec. III A, both null expansions of class A are written in the same form as Eq. (4.3) for any i . Thus, all type A1 surfaces have negative both null expansions.

2. Type A2 surface

In region II, we introduce the $(n+1)$ -surface of class A in which V and R satisfy the condition

$$\frac{dV}{dR} = \frac{a}{b - X^n}, \quad R = \rho, \quad (4.4)$$

where $X = V/R$, a and b are positive constants. We call this surface *type A2*. In this type, we find

$$V' = \frac{a}{b - X^n}, \quad V'' = \frac{naX^{n-1}}{R(b - X^n)^3} (X^{n+1} - bX + a), \quad R' = 1 \quad \text{and} \quad R'' = 0. \quad (4.5)$$

Substituting Eq. (4.5) and $m = \mu v^n$ into Eq. (3.9), we obtain the following both null expansions for type A2:

$$\vartheta_{(\pm)} = -N [(\mu a - n) \{naX^{n-1} + 3n(b-a)X^n + (2\mu a - n)X^{2n}\} + n^2(2b-a)(b-a)], \quad (4.6)$$

where

$$N = \frac{\sqrt{n}}{R\sqrt{2a}} \{n(2b-a) + 2X^n(\mu a - n)\}^{-3/2}. \quad (4.7)$$

To make both null expansions negative in Eq. (4.6) we impose the following sufficient conditions:

$$b > a, \quad (4.8)$$

$$\mu a > n. \quad (4.9)$$

Thus, if conditions (4.8) and (4.9) are satisfied, both null expansions of type A2 are negative for any i .

3. Type AB surface

In region III, we introduce two types of $(n+1)$ -surfaces. The first one is the type of class A which satisfies

$$V = \rho, \quad R = r_0 \quad (4.10)$$

where r_0 is the positive constant. We call this surface *type AB*. Both null expansions of type AB are

$$\vartheta_{(\pm)} = \frac{1}{\sqrt{2\Delta}} \frac{n}{R} \left(1 - \frac{M}{nR^n} \right), \quad (4.11)$$

where we have substituted $V' = 1$, $V'' = 0$, $R' = R'' = 0$ and $m = M$ into Eq. (3.9). Negative both null expansions are obtained, if and only if the condition $M > nR^n$ is satisfied. For convenience, we define $\gamma = nR^n/M$. Then, if and only if γ satisfies the condition

$$0 < \gamma < 1, \quad (4.12)$$

both null expansions (4.11) are negative.

4. Type B1 surface

In region III, we have introduced two types of $(n+1)$ -surfaces. The first one has been type AB. The second is the type of class B which is a capping disk defined by

$$\Theta_i^2 + \left(\frac{\rho}{r_0} - \sigma_0 \right)^2 = \frac{\pi^2}{4}, \quad V = \rho \quad (4.13)$$

where σ_0 is the positive constant. We call this surface *type B1*. Derivatives of type B1 are

$$V' = 1, \quad V'' = 0, \quad \Theta'_i = -\frac{\tilde{V}}{r_0} \left(\frac{\pi^2}{4} - \tilde{V}^2 \right)^{-1/2} \quad \text{and} \quad \Theta''_i = -\frac{\pi^2}{4r_0^2} \left(\frac{\pi^2}{4} - \tilde{V}^2 \right)^{-3/2}, \quad (4.14)$$

where $\tilde{V} = (V/r_0) - \sigma_0$. As calculated in Sec. III B, the expression of both null expansions of class B depends on i . Substituting Eq. (4.14) and $m = M$ into Eq. (3.15), we obtain both null expansions of type B1 for the case of $i = 1$ as follows:

$$\vartheta_{1(\pm)} = A_1 \left[\pm \frac{\pi^2}{4} - \sqrt{\tilde{\Delta}_1} \left\{ n \left(\frac{1}{\gamma} - 1 \right) \left(\frac{\pi^2}{4} - \tilde{V}^2 \right) + (n+1)\tilde{V}^2 \right\} \pm n\tilde{\Delta}_1 \sqrt{\frac{\pi^2}{4} - \tilde{V}^2} \cot \Theta_1 \right] \quad (4.15)$$

where $\tilde{\Delta}_1 = \Sigma \left\{ (\pi^2/4) - \tilde{V}^2 \right\} + \tilde{V}^2$ and $A_1 = \sqrt{\Sigma}/(\sqrt{2}r_0\tilde{\Delta}_1^{3/2})$. We obtain negative both null expansions, if the condition

$$\sqrt{\tilde{\Delta}_1} \left\{ n \left(\frac{1}{\gamma} - 1 \right) \left(\frac{\pi^2}{4} - \tilde{V}^2 \right) + (n+1)\tilde{V}^2 \right\} > \frac{\pi^2}{4} + n\tilde{\Delta}_1 \sqrt{\frac{\pi^2}{4} - \tilde{V}^2} \cot \Theta_1 \quad (4.16)$$

is satisfied. On the other hand, substituting Eq. (4.14) and $m = M$ into Eq. (3.16), we obtain both null expansions of type B1 for the case of $i \neq 1$ as follows:

$$\begin{aligned} \vartheta_{i(\pm)} = & A_i \left[\pm \frac{\pi^2}{4} \prod_{k=1}^{i-1} \sin \phi_k - \sqrt{\tilde{\Delta}_i} \left\{ n \left(\frac{1}{\gamma} - 1 \right) \left(\frac{\pi^2}{4} - \tilde{V}^2 \right) + (n+1)\tilde{V}^2 \prod_{k=1}^{i-1} \sin^2 \phi_k \right\} \right. \\ & \left. \pm n\tilde{\Delta}_i \sqrt{\frac{\pi^2}{4} - \tilde{V}^2} \cot \Theta_i \prod_{k=1}^{i-1} \frac{1}{\sin \phi_k} \right] \end{aligned} \quad (4.17)$$

where $\tilde{\Delta}_i = \Sigma \left(\pi^2/4 - \tilde{V}^2 \right) + \tilde{V}^2 \prod_{k=1}^{i-1} \sin^2 \phi_k$ and $A_i = \sqrt{\Sigma}/(\sqrt{2}r_0\tilde{\Delta}_i^{3/2})$. If the condition

$$\sqrt{\tilde{\Delta}_i} \left\{ n \left(\frac{1}{\gamma} - 1 \right) \left(\frac{\pi^2}{4} - \tilde{V}^2 \right) + (n+1)\tilde{V}^2 \prod_{k=1}^{i-1} \sin^2 \phi_k \right\} > \frac{\pi^2}{4} \prod_{k=1}^{i-1} \sin \phi_k + n\tilde{\Delta}_i \sqrt{\frac{\pi^2}{4} - \tilde{V}^2} \cot \Theta_i \prod_{k=1}^{i-1} \frac{1}{\sin \phi_k} \quad (4.18)$$

is satisfied, we obtain negative both null expansions. However, when we take the limit ϕ_k to 0, the second term on the right-hand side of Eq. (4.18) has infinitely large value. From this reason, we cannot obtain negative both null expansions of type B1 with $\phi_k = 0$ in the case of $i \neq 1$. To avoid this difficulty from now on we only focus on type B1 with $i = 1$.

B. Matching of four types of $(n+1)$ -surfaces

We shall match four types of $(n+1)$ -surfaces given in the previous section. Schematic figures of these surfaces are shown in Fig. 3. We match a type A1 surface with a type A2 surface on $v = 0$, the type A2 surface with a type AB

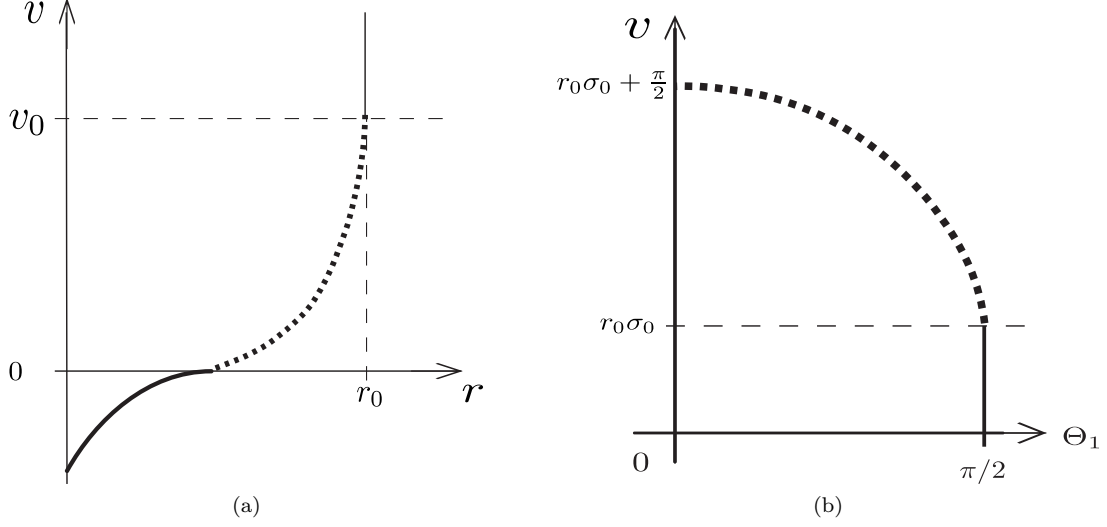


FIG. 3: Schematic figures of the matching surface. (a) Schematic figure of type A1, A2, and AB surfaces on the vr plane, where the thick solid line, the dotted line and the solid line are type A1, type A2 and type AB, respectively. (b) Schematic figure of type AB and type B1 on the $v\Theta_1$ plane, where the solid line and the dotted line are type AB and type B1, respectively.

surface on $v = v_0$, and the type AB surface with a type B1 surface on $v = r_0\sigma_0$ in region III.

1. Matching type A1 with type A2

We match the type A1 surface with the type A2 surface on $v = 0$. In order to obtain a smooth matching surface we match the derivative of both surfaces. From Eq. (4.2) we know that the derivative dv/dr of type A1 is positive and is less than one

$$0 < \frac{dv}{dr} < 1. \quad (4.19)$$

Thus, the derivative of type A2 must satisfy this condition (4.19) on $v = 0$. Substituting $v = 0$, i.e., $X = 0$ into Eq. (4.4), and imposing the condition (4.19) on it, we obtain the condition

$$0 < \frac{a}{b} < 1. \quad (4.20)$$

Note that this condition is written in the same form as the condition (4.8).

2. Matching type A2 with type AB

We match the type A2 surface with the type AB surface on $v = v_0$. Note that the derivative dv/dr of type AB is infinitely large

$$\frac{dv}{dr} \rightarrow \infty. \quad (4.21)$$

Choosing $b = X^n$ in Eq. (4.4), we can match the derivative of the type AB surface with that of type A2. Thus, in region II, X satisfies $0 < X^n < b$. On $v = v_0$ the type AB surface has the constant radius $r = r_0$. Substituting $r = r_0$ and $v = v_0$ into $b = X^n = (v/r)^n$, we express b as

$$b = \frac{n}{\mu\gamma}, \quad (4.22)$$

where $\gamma = nr_0^n/M$.

It should be noted that in the study of Bengtsson and Senovilla, they rewrote the derivative dV/dR as follows

$$R \frac{dX}{dR} = \frac{F(X)}{b - X} \quad (4.23)$$

where $F(X) = X^2 - bX + a$, and demanded $F(X) > 0$ to obtain the solution they wanted. In our study Eq. (4.23) becomes

$$R \frac{dX}{dR} = \frac{G(X)}{b - X^n} \quad (4.24)$$

where $G(X) = X^{n+1} - bX + a$. Similarly to Bengtsson and Senovilla's study, we shall impose $G(X) > 0$. The minimum value of $G(X)$ is given by

$$G(X_{\min}) = a - \frac{n}{(n+1)^{(n+1)/n}} b^{(n+1)/n}, \quad (4.25)$$

where $X_{\min} = (b/(n+1))^{1/n}$, since

$$\frac{dG(X_{\min})}{dX} = (n+1)X_{\min}^n - b = 0. \quad (4.26)$$

To get $G(X) > 0$, we impose the condition on parameters a and b from the following two cases: (a) $X_{\min} < b$ and $G(X_{\min}) > 0$, (b) $b < X_{\min}$ and $G(b) > 0$. Note that in the four-dimensional ($n = 1$) spacetime we get $X_{\min} = b/2$. Thus, we cannot use case (b) in this spacetime. Substituting $n = 1$ into $G(X_{\min}) > 0$, we obtain the condition as

$$a > \frac{b^2}{4}. \quad (4.27)$$

This condition is given in Bengtsson and Senovilla's study [5]. We show the allowed region of conditions (4.20) and (4.27) on the ab plane in Fig. 4(a). On the other hand, we shall consider the $(n+3)$ -dimensional spacetime with $n > 1$. In case (a), substituting $X_{\min} = (b/(n+1))^{1/n}$ into conditions $X_{\min} < b$ and $G(X_{\min}) > 0$, and after some calculation, we find

$$b > K, \quad \text{and} \quad a > n \left(\frac{b}{n+1} \right)^{(n+1)/n}, \quad (4.28)$$

where $K = (n+1)^{1/(1-n)}$. In case (b), we get

$$b < K \quad \text{and} \quad a > b^2 - b^{n+1}, \quad (4.29)$$

where we have rewritten conditions $b < X_{\min}$ and $G(b) > 0$. We also show the allowed region of conditions on parameters a and b as in Fig. 5(a). The allowed region of conditions (4.20) and (4.28) is shown by the dark-shaded area on the ab plane, while the pale-shaded area shows the allowed region of conditions (4.20) and (4.29). Note that in Fig. 5(a) we have substituted $n = 2$ into these conditions for reference.

3. Matching type AB with type B1

We match the type AB surface with the type B1 surface on $v = r_0\sigma_0$ in region III. From Fig. 3(b) the derivative $d\theta/dv$ of these surfaces vanishes on $v = r_0\sigma_0$. Thus, we can smoothly match these surfaces on $v = r_0\sigma_0$, i.e., $\tilde{V} = 0$ the condition (4.16) becomes

$$n\sqrt{\frac{2}{\gamma} - 1} \left(\frac{1}{\gamma} - 1 \right) > \frac{2}{\pi} \quad (4.30)$$

where we have substituted $\tilde{V} = 0$ into Eq. (4.16). Now, we shall find the upper bound on γ in Eq. (4.30). The solution of Eq. (4.30) is

$$\gamma < \gamma_0 = \frac{6}{6 + \Gamma}, \quad (4.31)$$

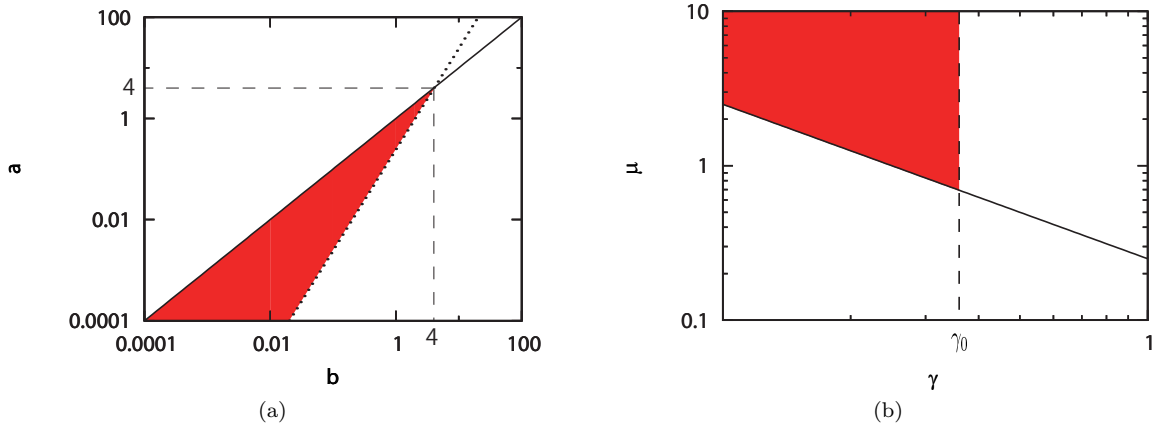


FIG. 4: (a) Schematic figure of the allowed region for conditions on the ab plane. The solid line and the dotted line are $a = b$ and $a = b^2/4$, respectively. The shaded area is the allowed region of conditions (4.20) and (4.27). (b) Schematic figure of the allowed region for conditions on the $\mu\gamma$ plane. The solid line denotes $\mu = 1/(4\gamma)$. γ_0 is the upper bound of γ imposed by the condition (4.30). The allowed region of conditions (4.30) and (4.36) is shown by the shaded area.

where γ_0 is the upper bound on γ ,

$$\Gamma = -1 + (E + F)^{1/3} + (E + F)^{-1/3}, \quad (4.32)$$

and

$$E = -1 + 54 \left(\frac{2}{n\pi} \right)^2, \quad F = 6 \sqrt{-3 \left(\frac{2}{n\pi} \right)^2 + 81 \left(\frac{2}{n\pi} \right)^4}. \quad (4.33)$$

When we choose $n = 1$, $\gamma_0 \simeq 0.6851$ which coincides with the upper bound on γ derived in Ref. [5]. For $n > 3$, F becomes a pure imaginary number. Then we can express Γ as

$$\Gamma = -1 + 2 \cos \left(\frac{\zeta}{3} \right), \quad (4.34)$$

where $\zeta = \arccos(E)$. ζ is the monotonically decreasing function of n . In the large n limit $\zeta = \pi$, while for $n = 4$, $\zeta \simeq 1.892$. Thus, γ_0 monotonically increases and approaches one in the large n limit. On the other hand, γ_0 gets a minimum value $\gamma_{\min} = 6/(6 + \Gamma_2) \simeq 0.7946$ on $n = 2$, where $\Gamma_2 = -1 + (E_2 + F_2)^{1/3} + (E_2 + F_2)^{-1/3}$, $E_2 = -1 + (54/\pi^2)$ and $F_2 = 6\sqrt{-3 + (81/\pi^2)}/\pi$. Note that the condition (4.30) on γ is stronger than the condition (4.16) on it.

Now we shall discuss the physical meaning of γ_0 . For this purpose, we introduce the following variable:

$$\bar{\gamma}_0 = \left(\frac{\gamma_0}{2} \right)^{1/n} = \frac{r_0}{r_g} \quad (4.35)$$

where $r_g = (2M/n)^{1/n}$ is the Schwarzschild-Tangherlini radius, and r_0 is the maximum radius of the trapped surface considered here. Thus, $\bar{\gamma}_0$ denotes the maximum radius of the trapped surface normalized by r_g for each dimension. We can find that $\bar{\gamma}_0$ also monotonically increases and also approaches one in the large n limit. Therefore, in the large n limit, the maximum radius of the trapped surface approaches r_g . Note that when we take the large n limit the vv -component in the $(n + 3)$ -dimensional Vaidya metric (2.1) does not approach zero. Instead, it approaches one in this limit.

C. Condition on the mass parameter

In previous sections, we have imposed several conditions on parameters to obtain a trapped surface. We shall combine these conditions and shall get the condition on the mass parameter μ .

In the four-dimensional case, from Fig. 4(a) we can see $0 < a < 4$ and $0 < b < 4$, respectively. Substituting Eq. (4.22) with $n = 1$ into $b < 4$, we get the condition on μ as

$$\mu > \frac{1}{4\gamma}. \quad (4.36)$$

In Fig. 4(b), we show the allowed region of conditions (4.30) and (4.36). From this figure we can understand that if the mass function satisfies $\mu > 1/(4\gamma_0) \simeq 0.3649$, we can obtain a trapped surface extended into region I. This result is given in Ref. [5].

Next, we shall discuss the $(n+3)$ -dimensional spacetime with $n > 1$. From the dark-shaded area in Fig. 5(a) we can see that a and b satisfy $L < a < N$ and $K < b < N$, respectively, where $K = (n+1)^{1/(1-n)}$, $L = n(n+1)^{(n+1)/(1-n)}$ and $N = (n+1)^{n+1}/n^n$. Combining $a < N$ with Eq. (4.9), we get the condition on μ as

$$\mu > \frac{n}{N}. \quad (4.37)$$

Substituting Eq. (4.22) into $K < b < N$, we also get the condition on μ as

$$\frac{n}{K\gamma} > \mu > \frac{n}{N\gamma}. \quad (4.38)$$

We combine conditions (4.30), (4.37) and (4.38), and show the allowed region of this combined condition in Fig. 5(b). In Fig. 5(b), the pale-shaded area is the allowed region of this combined condition, where we have substituted $n = 2$ into these conditions for reference. On the other hand, from the pale-shaded area in Fig. 5(a) we can see that a and b are bounded above by K . Combining $a < K$ with Eq. (4.9), we get the condition on μ as

$$\mu > \frac{n}{K}. \quad (4.39)$$

Substituting Eq. (4.22) into $b < K$, we also get the condition on μ as

$$\mu > \frac{n}{K\gamma}. \quad (4.40)$$

We also combine conditions (4.30), (4.39) and (4.40), and show the allowed region of this combined condition by the dark-shaded area in Fig. 5(b). Fig. 5(b) shows that μ is bounded below by $\mu = n/N$. Thus, at least if $\mu > n/(N\gamma)$ and $\gamma < \gamma_0$, we can construct a trapped surface extended into region I by the appropriate choice of a and b . n/N is a monotonically increasing function and it approaches $1/e$ in the large n limit. On the other hand, γ_0 is also the monotonically increasing function. γ_0 approaches one in the large n limit, while it has the minimum value γ_{\min} for $n = 2$. Thus, $1/\gamma_0$ gets the maximum value $1/\gamma_{\min} \simeq 1.258$ on $n = 2$. To put all the conditions together, we conclude that in the $(n+3)$ -dimensional self-similar Vaidya spacetime with $n > 1$ if the mass parameter μ is greater than $1/(\gamma_{\min}e) \simeq 0.4628$, we can construct a trapped surface extended into region I.

D. Naked singularity and the trapped surface

In Bengtsson and Senovilla's study, they investigated the four-dimensional self-similar Vaidya spacetime and showed that there is no naked singularity, if the spacetime has a trapped surface extended into region I [5]. How about this feature in the higher-dimensional spacetime? From this context, we shall discuss naked singularity in the $(n+3)$ -dimensional self-similar Vaidya spacetime with $n > 1$. In Sec. II we have mentioned that if and only if the mass parameter satisfies the condition (2.5), there exists naked singularity. On the other hand, in Sec. IV C we have shown the condition $\mu > 1/(\gamma_{\min}e)$ on the mass parameter so as to construct a trapped surface extended into region I. Comparing both conditions, we find

$$\left(\frac{1}{2}\right)^{n+1} < \frac{1}{\gamma_{\min}}, \quad \text{and} \quad \left(\frac{n}{n+1}\right)^{n+1} < \frac{1}{e}. \quad (4.41)$$

Thus, both conditions on the mass parameter are inconsistent with each other. Therefore, we conclude that if a trapped surface extended into region I can be constructed by the above discussion, there is no naked singularity in the $(n+3)$ -dimensional self-similar Vaidya spacetime.

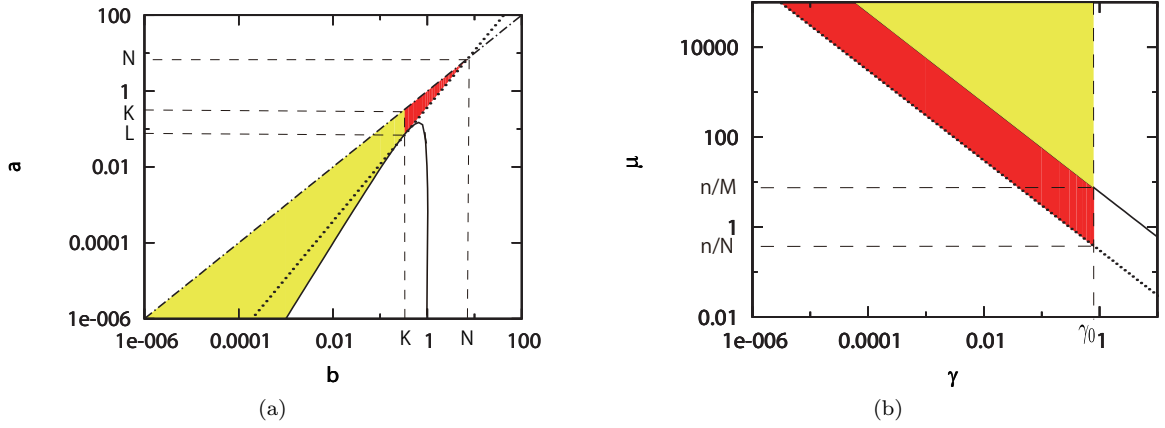


FIG. 5: (a) Schematic figure of the allowed region for conditions on the ab plane. the dotted-dashed line, the dotted line and the solid line are $a = b$, $a = n(b/n + 1)^{(n+1)/n}$ and $a = b^2 - b^{n+1}$, respectively. The dark-shaded area shows the allowed region of conditions (4.20) and (4.28), while the pale-shaded area shows the allowed region of conditions (4.20) and (4.29), where $K = (n + 1)^{1/(1-n)}$, $L = n(n + 1)^{(n+1)/(1-n)}$ and $N = (n + 1)^{n+1}/n^n$. Note that in this figure we have substituted $n = 2$ into these conditions for reference. (b) Schematic figure of the allowed region for conditions on $\mu\gamma$ plane, where the solid line and the dotted line are $\mu = n/(K\gamma)$ and $\mu = n/(N\gamma)$, respectively. The allowed region of conditions (4.30), (4.37) and (4.38) is shown by the dark-shaded area, while the pale-shaded area shows the allowed region of conditions (4.30), (4.39) and (4.40). γ_0 is the upper bound of γ imposed by the condition (4.30). Note that in this figure we have substituted $n = 2$ into these conditions for reference.

V. CONCLUSION AND DISCUSSIONS

We have investigated a trapped surface and naked singularity in the $(n + 3)$ -dimensional self-similar Vaidya space-time. A trapped surface is defined as the closed spacelike $(n + 1)$ -surface which has negative both null expansions. To construct a trapped surface we have introduced two classes of $(n + 1)$ -surfaces and have made both null expansions of these surfaces negative. We also have introduced four types of $(n + 1)$ -surfaces and matched smoothly the type A1 surface with the type A2 surface on $v = 0$, the type A2 surface with the type AB surface on $v = v_0$, and the type AB surface with the type B1 surface on $v = r_0\sigma_0$. To obtain the smooth matching and negative both null expansions we have imposed conditions on parameters. Putting these conditions together, we have got the condition on the mass parameter, i.e., we have got the lower limit $1/(\gamma_{\min}e) \simeq 0.4628$ on the mass parameter for $n > 1$. Therefore, we have shown that in the $(n + 3)$ -dimensional self-similar Vaidya spacetime for $n > 1$, if the mass parameter is greater than $1/(\gamma_{\min}e) \simeq 0.4628$, we get a trapped surface extended into region I. Moreover, we have found that the maximum radius of the trapped surface constructed in this study monotonically increases for n . If we take the limit n to infinity, the maximum radius of the trapped surface approaches the Schwarzschild-Tangherlini radius. Also, we have shown that there is no naked singularity, if the trapped surface given by the above construction exists in the $(n + 3)$ -dimensional self-similar Vaidya spacetime. These results are similar to those of Bengtsson and Senovilla in the four-dimensional case. However, the conditions are affected by the spacetime dimension, which can be seen in Figs. 4 and 5.

The trapped surface constructed in this study satisfies the condition $\gamma < 1$. This surface exists inside the black hole, because the event horizon satisfies the condition $\gamma = 2$. Although in the higher-dimensional spacetime we can have the large variety of $(n + 1)$ -surfaces, while we have introduced two classes of $(n + 1)$ -surfaces in this paper. Thus, if we introduce other kinds of $(n + 1)$ -surfaces and choose the appropriate matching for these surfaces, we can construct other kinds of trapped surfaces extended into region I.

Although Eardley's conjecture is considered only in four-dimensional spacetimes, applying this conjecture into the general dimension might be fruitful to define dynamical black holes in asymptotically flat higher-dimensional spacetimes.

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